SENSITIVITY ANALYSIS OF THE DISCRETE-TIME LMI BASED BOUNDED OUTPUT ENERGY PROBLEM

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Abstract: The bounded output energy problem is connected with obtaining a control law that ensures closed-loop stability, specified performance and minimizes the output energy. The specified control problem can be implicitly realized by the solutions Q, Y of a system of linear matrix inequalities (LMIs). The paper is concerned with obtaining linear perturbation bounds for the discrete-time LMI based bounded output energy problem, which are linear functions of the data perturbations. The sensitivity analysis of the perturbed matrix inequalities is considered in a similar manner as for perturbed matrix equations, after introducing a suitable right hand part, which is slightly perturbed. The proposed approach leads to tight linear perturbation bounds for the LMIs' solutions to the bounded output energy problem. Numerical example is also presented.

Key words: Perturbation analysis, bounded output energy problem, LMI based synthesis, Linear systems

1. Introduction

In many control problems, the design constraints have a simple reformulation in terms of linear matrix inequalities (LMIs). This is hardly surprising, given that LMIs are direct byproducts of Lyapunov based criteria, and that Lyapunov techniques play a central role in the analysis and control of linear systems, see [1,2] and the literature therein. The bounded output energy problem is a good illustration of this point.
The effectiveness of LMI approach remains valuable for several reasons. To begin with it is applicable to all plants without restrictions on infinite or pure imaginary invariant zeros. In addition LMI based design is practical and interesting thanks to the availability of efficient convex optimization algorithms [3] and software [4] plus the MATLAB package Yalmip and SeDuMi solver [5].
In this paper we propose an approach to perform linear sensitivity analysis of the LMI based bounded output energy problem via introducing a suitable right hand part in the considered matrix inequalities.

We use the following notations: $R^{m\times n}$ - the space of real $m\times n$ matrices; $R^n = R^{n\times 1}$; I_n - the identity $n\times n$ matrix; e_n - the unit $n\times 1$ vector; M^T - the transpose of M; $M^\dagger$ - the pseudo inverse of M; $\|M\|_2 = \sigma_{\max}(M)$ - the spectral norm of M, where $\sigma_{\max}(M)$ is the maximum singular value of M; $vec(M)\in R^{mn}$ - the column-wise vector representation of $M\in R^{m\times n}$; $\Pi_{m,n}\in R^{mn\times mn}$ - the vec-permutation matrix, such that $vec(M^T)=\Pi_{m,n}vec(M)$; $M\otimes P$ - the Kroneker product of the matrices M and P. The notation ":=" stands for "equal by definition".
The remainder of the paper is organized as follows. In Section 2 we shortly present the problem set up and objective. Section 3 describes the performed linear sensitivity analysis of the LMI based bounded output energy problem. Section 4 presents a numerical example before we conclude in Section 5 with some final remarks.

2. Problem Set up and Objective

Consider the linear discrete-time system

$$\begin{aligned} x(k+1) &= Ax(k)+Bu(k), \\ y(k) &= Cx(k) \end{aligned} \qquad (1)$$

where $x(k)\in R^n, u(k)\in R^m$, and $y(k)\in R^r$ are the system state, input and output vectors respectively, and A, B, C are constant matrices of compatible size.
Bounded output energy problem means for a given initial state $x(0)$ to find a control law, which minimizes the output energy

$\sum_0^{\infty} y^T(k)y(k)$. It is also necessary to find a quadratic Lyapunov function $V[x(k)] = x^T(k)Px(k), \quad P > 0$ such that $V[x(k+1)] - V[x(k)] < -y^T(k)y(k)$.

In order to solve the bounded output energy problem and to ensure closed-loop stability and specified performance it is necessary to design a state-feedback control *u(k)=Kx(k)*. Further we will skip the dependence of index "*k*".

We consider an LMI approach to solve the bounded output energy problem, as stated in [1].

$$x^T[(A+BK)^T P(A+BK)-P]x < -y^T y, P>0. \quad (2)$$

Unfortunately inequality (2) is not an LMI with respect to the decision variables *K* and *P*. That is why we perform the substitution $Q = P^{-1}, Q > 0$ and $Y = KP^{-1}$ to obtain the following system of LMIs:

$$Q-(AQ+BY)^T Q^{-1}(AQ+BY)-(CQ)^T CQ>0 \quad (3)$$

To transform LMI (3) we apply Schur complement argument [6] to obtain the following inequality:

$$\begin{bmatrix} -Q & (AQ+BY)^T & QC^T \\ (AQ+BY) & -Q & 0 \\ CQ & 0 & -I \end{bmatrix} < 0, \quad (4)$$
$$Q > 0$$

with respect to the variables *Q* and Y.

The main objective of the paper is to perform a linear sensitivity analysis of the LMI system (4) needed to solve the bounded output energy problem.

Suppose that the matrices *A, B, C* are subject to perturbations $\Delta A, \Delta B, \Delta C$ and assume that they do not change the sign of the LMI system (4). The sensitivity analysis of the discrete-time LMI based bounded output energy problem is aimed at determining perturbation bounds of the LMIs (4) as functions of the perturbations in the data *A, B, C.*

3. Linear Sensitivity Analysis

We perform sensitivity analysis of the LMI (4) for the discrete-time system (1).

$$\begin{bmatrix} -(Q+\Delta Q) & ABQY^T & qQC^T \\ ABQY & -(Q+\Delta Q) & 0 \\ qQC & 0 & -I \end{bmatrix} < 0 \quad (5)$$

where
$ABQY^T = (Q+\Delta Q)(A+\Delta A)^T + (Y+\Delta Y)^T(B+\Delta B)^T$
$ABQY = (A+\Delta A)(Q+\Delta Q) + (B+\Delta B)(Y+\Delta Y)$,
$qQC^T = (Q+\Delta Q)(C+\Delta C)^T$,
$qCQ = (C+\Delta C)(Q+\Delta Q)$. We have to study the effect of the perturbations $\Delta A, \Delta B, \Delta C$ on the perturbed LMI solutions $Q^* + \Delta Q$ and $Y^* + \Delta Y$, where Q^*, Y^* and $\Delta Q, \Delta Y$ are the nominal solution of the inequality (4) and the perturbations, respectively. The essence of our approach is to perform sensitivity analysis of the inequality (4) in a similar manner as for a proper matrix equation after introducing a suitable right hand part, which is slightly perturbed. Thus for LMI (5) we have:

$$\begin{bmatrix} -(Q^*+\Delta Q) & ABQY^{*T} & qQC^{*T} \\ ABQY^* & -(Q^*+\Delta Q) & 0 \\ qQC^* & 0 & -I \end{bmatrix} = N^* + \Delta N_1 < 0 \quad (6)$$

where $ABQY^{*T} = (Q^*+\Delta Q)(A+\Delta A)^T + (Y^*+\Delta Y)^T(B+\Delta B)^T$,
$ABQY^* = (A+\Delta A)(Q^*+\Delta Q) + (B+\Delta B)(Y^*+\Delta Y)$
$qQC^{*T} = (Q^*+\Delta Q)(C+\Delta C)^T$,
$qCQ^* = (C+\Delta C)(Q^*+\Delta Q)$ and N^* is obtained using the nominal LMI

$$\begin{bmatrix} -Q^* & Q^*A^T + Y^{*T}B^T & Q^*C^T \\ AQ^*+BY^* & -Q^* & 0 \\ CQ^* & 0 & -I \end{bmatrix} = N^* < 0. \quad (7)$$

The matrix ΔN_1 is due to the data and closed-loop performance perturbations, the rounding errors and the sensitivity of the interior point method that is used to solve the LMIs.

Using the relation (7) the perturbed equation (6) may be written as

$$\Delta_Q + \Omega_Q = \Delta N_1, \qquad (8)$$

where

$$\Delta_Q = \begin{bmatrix} -\Delta Q & \Delta Q A^T & \Delta Q C^T \\ A \Delta Q & -\Delta Q & 0 \\ C \Delta Q & 0 & 0 \end{bmatrix},$$

$$\Omega_Q = \begin{bmatrix} 0 & Q^* \Delta A^T + \Delta Y^T B^T + Y^{*T} \Delta B^T & Q^* \Delta C^T \\ \Delta A Q^* + B \Delta Y + \Delta B Y^* & 0 & 0 \\ \Delta C Q^* & 0 & 0 \end{bmatrix}.$$

Here the terms of second and higher order are neglected. The relation (8) may be written in a vector form as

$$vec(\Delta_Q) + vec(\Omega_Q) = vec(\Delta N_1), \qquad (9)$$

Where

$$vec(\Delta_Q) = [-I, A \otimes I, C \otimes I, I \otimes A, -I, I \otimes C, 0, 0]^T vec(\Delta Q) = T \Delta q,$$

$$vec(\Omega_Q) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ (I \otimes Q^*)\Pi_{n^2} & (B \otimes I)\Pi_{n \times m} & (I \otimes Y^{*T})\Pi_{m^2} & 0 \\ 0 & 0 & 0 & (I \otimes Q^*)\Pi_{n^2} \\ (Q^* \otimes I) & (I \otimes B) & (Y^* \otimes I) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (Q^* \otimes I) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} vec(\Delta A) \\ vec(\Delta Q) \\ vec(\Delta B) \\ vec(\Delta C) \end{bmatrix} = [T_{t1}, T_{t2}, T_{t3}, T_{t4}] \Delta_{aybc} = T_t \Delta_{aybc}.$$

Further we obtain the expression

$$T \Delta q + T_{t1} vec(\Delta A) + T_{t2} vec(\Delta Y) + T_{t3} vec(\Delta B) + T_{t4} vec(\Delta C) = = vec(\Delta N_1). \qquad (10)$$

Finally the relative perturbation bound for the solution Q^* of the LMI (4) has the form

$$\frac{\|\Delta q\|_2}{\|vec(Q^*)\|_2} \le \frac{1}{\|vec(Q^*)\|_2} \left(T_1 \frac{\|vec(\Delta A)\|_2}{\|vec(A)\|_2} + T_2 \frac{\|vec(\Delta Y)\|_2}{\|vec(Y^*)\|_2} + T_3 \frac{\|vec(\Delta B)\|_2}{\|vec(B)\|_2} \right) + \frac{1}{\|vec(Q^*)\|_2} \left(T_4 \frac{\|vec(\Delta C)\|_2}{\|vec(C)\|_2} + N_1 \frac{\|vec(\Delta N_1)\|_2}{\|vec(N^*)\|_2} \right) \qquad (11)$$

where

$$\frac{T_1}{\|vec(Q^*)\|_2} = \frac{\|T^\dagger\|_2 \|T_{t1}\|_2 \|vec(A)\|_2}{\|vec(Q^*)\|_2}, \quad \frac{T_2}{\|vec(Q^*)\|_2} = \frac{\|T^\dagger\|_2 \|T_{t2}\|_2 \|vec(Y^*)\|_2}{\|vec(Q^*)\|_2},$$

$$\frac{T_3}{\|vec(Q^*)\|_2} = \frac{\|T^\dagger\|_2 \|T_{t3}\|_2 \|vec(B)\|_2}{\|vec(Q^*)\|_2}, \quad \frac{T_4}{\|vec(Q^*)\|_2} = \frac{\|T^\dagger\|_2 \|T_{t4}\|_2 \|vec(C)\|_2}{\|vec(Q^*)\|_2},$$

$$\frac{N_1}{\|vec(Q^*)\|_2} = \frac{\|T^\dagger\|_2 \|vec(N^*)\|_2}{\|vec(Q^*)\|_2}.$$

may be considered as individual relative condition numbers of the LMI (4) with respect to the perturbations $\Delta A, \Delta B, \Delta C$ and ΔY.

In a similar way the relative perturbation bounds for the solution Y^* of the LMI (4) may be obtained using the following expression

$$\Delta_Y + \Omega_Y = \Delta N_2, \qquad (12)$$

where

$$\Delta_Y = \begin{bmatrix} 0 & \Delta Y^T B^T & 0 \\ B \Delta Y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Omega_Y = \begin{bmatrix} -\Delta Q & \Delta Q A^T + Q^* \Delta A^T + Y^{*T} \Delta B^T & \Delta Q C^T + Q^* \Delta C^T \\ A \Delta Q + \Delta A Q^* + \Delta B Y^* & -\Delta Q & 0 \\ C \Delta Q + \Delta C Q^* & 0 & 0 \end{bmatrix}$$

.

Here the terms of second and higher order are neglected. The relation (12) may be written in a vector form as

$$vec(\Delta_Y)+vec(\Omega_Y)=vec(\Delta N_2), \qquad (13)$$

where

$$vec(\Delta_Y)=[0,(B\otimes I)\Pi_{n\times m},0,(I\otimes B),0,0,0,0,0]^T vec(\Delta Y)$$
$$=W\Delta q,$$

$$vec(\Omega_Y)=\begin{bmatrix} 0 & -I & 0 & 0\\ (I\otimes Q^*)\Pi_{n^2} & (A\otimes I) & (I\otimes Y^{*T})\Pi_{nm} & 0\\ 0 & (C\otimes I) & 0 & (I\otimes Q^*)\Pi_{n^2}\\ (Q^*\otimes I) & (I\otimes A) & (Y^*\otimes I) & 0\\ 0 & -I & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & (I\otimes C) & 0 & (Q^*\otimes I)\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\times\begin{bmatrix} vec(\Delta A)\\ vec(\Delta Y)\\ vec(\Delta B)\\ vec(\Delta C)\end{bmatrix}=[W_{t1},W_{t2},W_{t3},W_{t4}]\Delta_{aybc}=W_t\Delta_{aybc}.$$

Further we obtain the expression

$$W\Delta y+W_{t1}vec(\Delta A)+W_{t2}vec(\Delta Q)+W_{t3}vec(\Delta B)+W_{t4}vec(\Delta C)=$$
$$=vec(\Delta N_2). \qquad (14)$$

Finally the relative perturbation bound for the solution Y^* of the LMI (4) has the form

$$\frac{\|\Delta y\|_2}{\|vec(Y^*)\|_2}\le\frac{1}{\|vec(Y^*)\|_2}\left(W_1\frac{\|vec(\Delta A)\|_2}{\|vec(A)\|_2}+W_2\frac{\|vec(\Delta Q)\|_2}{\|vec(Q^*)\|_2}+W_3\frac{\|vec(\Delta B)\|_2}{\|vec(B)\|_2}\right)$$
$$+\frac{1}{\|vec(Y^*)\|_2}\left(W_4\frac{\|vec(\Delta C)\|_2}{\|vec(C)\|_2}+N_2\frac{\|vec(\Delta N_2)\|_2}{\|vec(N^*)\|_2}\right) \qquad (15)$$

where

$$\frac{W_1}{\|vec(Y^*)\|_2}=\frac{\|W^\dagger\|_2\|W_{t1}\|_2\|vec(A)\|_2}{\|vec(Y^*)\|_2},\ \frac{W_2}{\|vec(Y^*)\|_2}=\frac{\|W^\dagger\|_2\|W_{t2}\|_2\|vec(Q^*)\|_2}{\|vec(Y^*)\|_2},$$
$$\frac{W_3}{\|vec(Y^*)\|_2}=\frac{\|W^\dagger\|_2\|W_{t3}\|_2\|vec(B)\|_2}{\|vec(Y^*)\|_2},\ \frac{W_4}{\|vec(Y^*)\|_2}=\frac{\|W^\dagger\|_2\|W_{t4}\|_2\|vec(C)\|_2}{\|vec(Y^*)\|_2},$$
$$\frac{N_2}{\|vec(Y^*)\|_2}=\frac{\|W^\dagger\|_2\|vec(N^*)\|_2}{\|vec(Y^*)\|_2}.$$

may be considered as individual relative condition numbers of the LMI (4) with respect to the perturbations $\Delta A, \Delta B, \Delta C$ and ΔQ .

4. Numerical Examples

Consider the discrete-time system (1), where

$$A=\begin{bmatrix} 1 & 0 & 0.01 & 0\\ 0 & 1.0002 & -0.0001 & 0.01\\ 0 & 0.0065 & 0.9939 & 0\\ 0 & 0.0316 & -0.0185 & 1\end{bmatrix}, B=\begin{bmatrix} 0.0001\\ -0.0002\\ 0.0122\\ -0.0372\end{bmatrix},$$
$$C=\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}, D=\begin{bmatrix} 0\\ 0\end{bmatrix}.$$

The perturbations in the system matrices of the discrete-time system are chosen as

$$\Delta A=A\times 10^{-i}, \Delta B=B\times 10^{-i}, \Delta C=C\times 10^{-i},$$
$$\Delta N_1=N^*\times 10^{-i}, \Delta N_2=N^*\times 10^{-i},$$
$$\Delta Q^*=Q^*\times 10^{-i}, \Delta Y=Y^*\times 10^{-i} \text{ for } i=8,7,\ldots,4.$$

The perturbed solutions $Q^*+\Delta Q$ and $Y^*+\Delta Y$ are computed based on the method derived in [7] and using the software [4]. The relative perturbation bounds for the solutions Q^* and Y^* of the LMIs (4) are obtained by the linear bounds (11) and (15), respectively.

The results obtained for different values of i are shown in the following table

i	$\frac{\|\| \Delta q \|\|_2}{\|\| vec(Q^*) \|\|_2}$	Bound (11)	$\frac{\|\| \Delta y \|\|_2}{\|\| vec(Y^*) \|\|_2}$	Bound (15)
8	$1.75*10^{-8}$	$5.92*10^{-8}$	$7.40*10^{-8}$	$2.81*10^{-7}$
7	$1.75*10^{-7}$	$5.92*10^{-7}$	$7.40*10^{-7}$	$2.81*10^{-6}$
6	$1.75*10^{-6}$	$5.92*10^{-6}$	$7.40*10^{-6}$	$2.81*10^{-5}$
5	$1.75*10^{-5}$	$5.92*10^{-5}$	$7.40*10^{-5}$	$2.81*10^{-4}$
4	$1.75*10^{-4}$	$5.92*10^{-4}$	$7.40*10^{-4}$	$2.81*10^{-3}$

The obtained perturbation bounds (11) and (15), based on the presented solution approach, are close to the real relative perturbation bounds $\frac{|| \Delta q ||_2}{|| vec(Q^*) ||_2}$ and $\frac{|| \Delta y ||_2}{|| vec(Y^*) ||_2}$, thus they are good in sense that they are tight.

5. Conclusions

The linear sensitivity analysis of the discrete-time LMI based bounded output energy problem has been studied. Tight perturbation bounds, which are linear functions of the data perturbations, have been obtained for the matrix inequalities determining the problem solution. Based on these results we have presented numerical examples to explicitly reveal the performance and applicability of the proposed approach to analyze the sensitivity of the discrete-time LMI based bounded output energy problem.

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